Green index and finiteness conditions for semigroups

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Abstract

Let S be a semigroup and let T be a subsemigroup of S. Then T acts on S by left- and by right multiplication. If the complement $S \setminus T$ has finitely many strong orbits by both these actions we say that T has finite Green index in S. This notion of finite index encompasses subgroups of finite index in groups, and also subsemigroups of finite Rees index (complement). Therefore, the question of S and T inheriting various finiteness conditions from each other arises. In this paper we consider and resolve this question for the following finiteness conditions: finiteness, residual finiteness, local finiteness, periodicity, having finitely many right ideals, and having finitely many idempotents.

Key words: Finiteness conditions, index, residual finiteness, local finiteness.

1 Introduction

One of the most fundamental concepts in combinatorial group theory is the notion of index. It may be thought of as providing a way of measuring the difference between a group and a subgroup. In this sense we think of the finite index subgroups as only differing from the group by a finite amount. This is reflected in many theorems showing that groups are similar to their finite index subgroups, in terms of the combinatorial and algebraic properties that they share. For example, the properties of finiteness, being finitely generated, finite presentability, having a soluble word problem, periodicity, local finiteness, and residual finiteness are all known to be preserved by taking finite index subgroups and under taking finite index extensions. Questions relating to finite index subgroups and extensions continue to receive a lot of attention; see [19], [21] and [25] for example.

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Given a semigroup S and a subsemigroup T, the *Rees* index of T in S is defined to be the cardinality of the complement $S \setminus T$. The study of Rees index was initiated by Jura in [11], [12], and [13], and then developed and extended considerably in the papers [22], [24], [8], [27] and [17] . In this body of work many results were proved showing the preservation of finiteness conditions under taking finite Rees index substructures and finite Rees index extensions. In particular, all of the properties listed in the paragraph above were shown to be preserved. The main result of [22] is that finite presentability is preserved under finite Rees index substructures. This result was generalized in [6] to the so called finite boundary substructures. It is important to observe, however, that although there is this strong parallel between the group theoretic index and Rees index, the latter does not, in any sense, generalize the former, since an infinite group cannot have any proper finite Rees index sub(semi)groups.

A natural question arising from this observation is whether there is some unifying framework which would encompass both these notions. An attempt to develop one such a notion, called syntactic index, was made in [24]. It does provide a common generalization of subgroup and Rees indices, but it is not strong enough for any interesting theorems about preservation of properties to hold (see [24, Theorem 3.5]).

In this article we will introduce a new notion of index for subsemigroups. The definition is based on a generalization of the important structural concept known as Green's relations. Due to this connection we will name this new notion the *Green index* of a subsemigroup. This new concept provides a common generalization of subgroup and Rees indices. We will show that Green index is strong enough to prove common generalizations of both finite index subgroup and finite Rees index results. The idea, roughly speaking, is that the multiplication actions of T on S partition $S \setminus T$ in a natural way, into sets that we call (T-relative) $\mathcal{H}$-*classes*. We say that T has finite index if $S \setminus T$ is a union of finitely many $\mathcal{H}$-classes. Associated to each of these $\mathcal{H}$-classes is a group, which we call the *Schützenberger group* of that $\mathcal{H}$-class. Our theorems show how the properties of S are related to those of T and the finitely many Schützenberger groups. Related ideas may be found in [23] where, amongst other things, it is proved that if a semigroup S has only finitely many right and left ideals, then S is finitely presented if and only if all of its Schützenberger groups are finitely presented. A similar approach to subgroups of monoids was also considered in [10].

Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ were introduced in [7]; these equivalence relations classify the elements of a semigroup in terms of the principal ideals that they generate. Since their introduction they have played a central role in the development of the structure theory of semigroups. Our interest here will only be in the relations $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. In a semigroup S, two elements $x, y \in S$ are said to be $\mathcal{R}$-related if and only if they generate the same principal right ideal, i.e. $x\mathcal{R}y$ if and only if $xS^1 = yS^1$. Dually, $x\mathcal{L}y$ if and only if they generate the same principal left ideal, i.e. $S^1x = S^1y$. (Throughout this paper, S^1 will stand for the semigroup S with an identity element $1_S = 1 \notin S$ adjoined to it. This notation will extend to subsets of S, i.e. $X^1 = X \cup \{1\}$.)

Since their introduction various generalizations of Green's relations have been proposed and investigated; see [1], [18], [20] and [3] for example. In [26] Wallace introduced the idea of ideals and Green's relations taken relative to a subsemigroup T. In that paper Wallace showed that many of the classical results of Green carry across to this more general setting.

Let S be a semigroup and let T be a subsemigroup of S. For $u, v \in S$ define:

$$u\mathcal{R}^T v \;\Leftrightarrow\; uT^1 = vT^1, \quad u\mathcal{L}^T v \;\Leftrightarrow\; T^1u = T^1v,$$

and $\mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T$. Each of these relations is an equivalence relation on S; their equivalence classes are called the (T-)*relative* $\mathcal{R}$-, $\mathcal{L}$-, and $\mathcal{H}$-classes, respectively. Note that since T is a subsemigroup, the relative $\mathcal{R}$-, $\mathcal{L}$-, and $\mathcal{H}$-classes respect the partition $S = T \cup U$, where $U = S \setminus T$. In other words, each of these relations is contained in $(U \times U) \cup (T \times T)$, and each of U and T is a union of $\mathcal{R}^T$-classes, $\mathcal{L}^T$-classes and $\mathcal{H}^T$-classes.

Definition 1. Let S be a semigroup, let T be a subsemigroup of S, and let $U = S \setminus T$. The *Green index* of T in S is $[S : T]_G = |U/\mathcal{H}^T| + 1$.

Thus, a subsemigroup has finite Green index if its complement has only finitely many $\mathcal{H}^T$-classes (and hence also finitely many $\mathcal{R}^T$- and $\mathcal{L}^T$-classes).

For each T-relative $\mathcal{H}$-class H fix $h \in H$, let $\mathrm{Stab}(H) = \{t \in T^1 : ht \in H\}$ (the *stabilizer* of H in T), and define an equivalence $\sigma = \sigma(H)$ on $\mathrm{Stab}(H)$ by $(x, y) \in \sigma$ if and only if $hx = hy$ for all $h \in H$. Then σ is a congruence on $\mathrm{Stab}(H)$ and $\mathrm{Stab}(H)/\sigma$ is a group.

Definition 2. The group $\Gamma(H) = \mathrm{Stab}(H)/\sigma$ is called the *Schützenberger group* of H.

The main results of the paper are summarized in the following theorem.

Theorem 3. *Let S be a semigroup and let T be a subsemigroup of S with finite Green index. Let Γ_i $(i \in I)$ be the Schützenberger groups of the T-relative $\mathcal{H}$-classes of the complement $S \setminus T$. Then the following hold:*

(I) S is locally finite if and only if T is locally finite, in which case every group Γ_i is locally finite;
(II) S is periodic if and only if T is periodic, in which case every group Γ_i is periodic;

(III) S has finitely many right ideals if and only if T has finitely many right ideals (and the dual result for left ideals).

Moreover we have:

(IV) S is residually finite if and only if T and Γ_i ($i \in I$) are all residually finite.

In a subsequent paper we will consider the question of the preservation of the properties of being finitely generated and of being finitely presented.

We note that all of the finiteness conditions mentioned above are known to be preserved under taking finite index subgroups and taking finite index extensions, and also finite Rees index subsemigroups and extensions. Theorem 3 gives a common generalization of all of those results. This is obvious for the first three conditions of the theorem. The fact that the result for residual finiteness may be applied to both finite index subgroups and finite Rees index subsemigroups follows from Corollaries 28 and 29.

The structure of the paper is as follows. In §2 we discuss basic properties of Green index. Then in §3 we consider the properties of local finiteness, periodicity, and having finitely many right ideals, showing that each is preserved by finite Green index substructures and extensions. In §4 we prove the corresponding result for the property of residual finiteness. The relationship between Green index and syntactic index is determined in §5. Finally, in §6 we discuss a number of applications of our results.

2 Basic properties

2.1 Relative Green's relations

Throughout this paper S will be a semigroup, T will be a subsemigroup of S, and Green's relations in S will always be taken relative to T, unless otherwise stated. That is to say, we will write $x\mathcal{R}y$ to mean that $xT^1 = yT^1$ rather than $xS^1 = yS^1$. On the few occasions that we need to refer to Green's $\mathcal{R}$ relation in S we will write $\mathcal{R}^S$. The same goes for the relations $\mathcal{L}$ and $\mathcal{H}$. Some basic facts about the behaviour of relative Green's relations are summarized below.

Proposition 4. *Let S be a semigroup and let T be a subsemigroup of S.*

(i) The relation $\mathcal{R}$ is a left congruence on S, and $\mathcal{L}$ is a right congruence.
(ii) For each $\mathcal{H}$-class H either $H^2 \cap H = \varnothing$, or $H^2 \cap H = H$ in which case H is a subgroup of S.

(iii) *Let $u, v \in S$ be such that $u\mathcal{R}v$, and let $p, q \in T$ such that $up = v$ and $vq = u$. Then the mapping ρ_p given by $x \mapsto xp$ is an $\mathcal{R}$-class preserving bijection from L_u to L_v while the mapping ρ_q given by $x \mapsto xq$ is an $\mathcal{R}$-class preserving bijection from L_v to L_u, and is the inverse of the mapping ρ_p.*

Proof. Part (i) is obvious from the definition. Part (ii) is proved in [26, Section 2,(2.5)]. Part (iii) is proved in [26, Section 2,(2.1)]. □

2.2 Schützenberger groups

In Section 1 we saw how one can associate a group $\Gamma(H)$ to every $\mathcal{H}$-class H. The proofs of assertions made there, as well as of those listed below, are well known in the classical case of $T = S$; see [14, Section 2.3] for example. In each case, generalizing to arbitrary subsemigroups $T \leq S$ is a simple exercise; see [26] for more details.

Proposition 5. *Let S be a semigroup, let T be a subsemigroup of S, let H be an $\mathcal{H}$-class of S, and let $h \in H$ be an arbitrary element. Then:*

(i) $\mathrm{Stab}(H) = \{t \in T^1 : ht \in H\}$.
(ii) $\sigma(H) = \{(u, v) \in \mathrm{Stab}(H) \times \mathrm{Stab}(H) : hu = hv\}$.
(iii) $H = h\mathrm{Stab}(H)$.
(iv) *If H and H' belong to the same $\mathcal{L}^T$-class of S then* $\mathrm{Stab}(H) = \mathrm{Stab}(H')$.
(v) $\Gamma(H)$ *acts regularly on H. In particular* $|\Gamma(H)| = |H|$.
(vi) *If H_1 is an $\mathcal{H}$-class of S belonging to the same $\mathcal{R}$-class (or to the same $\mathcal{L}$-class) as H then* $\Gamma(H_1) \cong \Gamma(H)$.
(vii) *If H contains an idempotent then* $\Gamma(H) \cong H$.

2.3 Green, group and Rees indices

If $S \setminus T$ is finite, then, of course, it contains only finitely many $\mathcal{H}$-classes.

Proposition 6. *If a subsemigroup has finite Rees index then it has finite Green index.*

If T happens to be an ideal, then all $\mathcal{H}$-classes of $S \setminus T$ are singletons.

Proposition 7. *An ideal has finite Green index if and only if it has finite Rees index.*

If S is a group, and T a subgroup of S, then the $\mathcal{R}$-classes are the left cosets of

T, the $\mathcal{L}$-classes are the right cosets of T, and the $\mathcal{H}$-classes are the non-empty intersections of left and right cosets.

Proposition 8. *A subgroup of a group has finite Green index if and only if it has finite (group) index.*

2.4 A structural characterisation

We now prove a result which shows how each $\mathcal{R}^S$-class divides into $\mathcal{R}^T$-classes.

Proposition 9. *Let S be a semigroup and let T be a subsemigroup of S with finite Green index. Then each $\mathcal{R}^S$-class of S is a union of finitely many $\mathcal{R}^T$-classes.*

Proof. We may suppose that $S \setminus T \neq \varnothing$ since when $S = T$ the result holds trivially. Let R^S be an $\mathcal{R}^S$-class of S. Suppose, for the sake of a contradiction, that R^S is a union of infinitely many $\mathcal{R}^T$-classes. Since $S \setminus T$ has only finitely many $\mathcal{R}^T$-classes it follows that $R^S \cap T$ is non-empty and contains infinitely many $\mathcal{R}^T$-classes.

Claim 1. *For an arbitrary $t_1 \in R^S \cap T$ there is $x \in T$ such that $t_1 x \in R^S \cap T$ while for all $y \in T$ we have $t_1 x y \neq t_1$.*

Proof. By definition we have $R^S \subseteq t_1 S^1$. Let $u, v \in S \setminus T$ with $u \mathcal{R}^T v$. Since $\mathcal{R}^T$ is a left congruence $u \mathcal{R}^T v$ implies $t_1 u \mathcal{R}^T t_1 v$. It follows, since the Green index is finite, that $t_1(S \setminus T) \cap T$ intersects only finitely many of the infinitely many $\mathcal{R}^T$-classes contained in $R^S \cap T$. Therefore there are infinitely many $\mathcal{R}^T$-classes in $R^S \cap T$ that can only be reached from t_1 by right multiplication by elements from T. In particular we can find an $\mathcal{R}^T$-class R' in $R^S \cap T$ different from that of t_1 itself, and element $x \in T$ such that $t_1 x \in R'$. Since t_1 and $t_1 x$ do not belong to the same $\mathcal{R}^T$-class, and $x \in T$, we know that there is no element $y \in T$ satisfying $t_1 x y = t_1$. $\square$

Claim 2. *There exists a sequence of elements $t_1, t_2, t_3, \ldots \in R^S \cap T$ such that*

$$t_1 T^1 \supsetneq t_2 T^1 \supsetneq t_3 T^1 \supsetneq \ldots.$$

Proof. If t_1 and x are as in Claim 1, then letting $t_2 = t_1 x$ we clearly have $t_1 T^1 \supsetneq t_2 T^1$. But as t_1 is arbitrary we can continue this process, yielding an infinite sequence of elements as required. $\square$

Now, given a sequence as above, for all i choose $x_i \in S \setminus T$ so that $t_{i+1} x_i = t_i$. This is possible since these elements all belong to a single $\mathcal{R}^S$-class of S.

Note that for all $j \geq i \geq 1$ we have $t_{j+1}x_jx_{j-1}\dots x_i = t_i$, which implies $x_jx_{j-1}\dots x_i \in S \setminus T$ because $t_iT^1 \supsetneq t_{j+1}T^1$. Since the Green index is finite there exist $l > j > i$ such that

$$x_lx_{l-1}\dots x_j \mathcal{R}^T x_lx_{l-1}\dots x_jx_{j-1}\dots x_i.$$

It follows that there is some $t \in T$ satisfying $x_lx_{l-1}\dots x_i = x_lx_{l-1}\dots x_jt$ and therefore

$$t_i = t_{l+1}x_lx_{l-1}\dots x_i = t_{l+1}x_lx_{l-1}\dots x_jt = t_jt.$$

This is a contradiction with $t_iT^1 \supsetneq t_jT^1$, and the proof is complete. $\square$

Clearly there is also a dual result for $\mathcal{L}$-classes. The following result describes what finite Green index means in terms of (global) Green's relations.

Proposition 10. *Let S be a semigroup and let T be a subsemigroup of S. Then T has finite Green index in S if and only if the following conditions hold:*

(i) T contains all but finitely many of the $\mathcal{H}^S$-classes of S;
(ii) Each $\mathcal{H}^S$-class of S is a union of finitely many $\mathcal{H}^T$-classes.

Proof. Suppose that (i) and (ii) both hold. Then the total number of $\mathcal{H}^S$-classes intersecting $S \setminus T$ is finite by (i). Also, by (ii) each such intersection is a union of finitely many $\mathcal{H}^T$-classes. Therefore $S \setminus T$ has finitely many $\mathcal{H}^T$-classes and so T has finite Green index.

Conversely suppose that T has finite Green index. Then (i) holds since each $\mathcal{H}^S$-class is a union of $\mathcal{H}^T$-classes so the complement $S \setminus T$ can only have non-empty intersection with finitely many $\mathcal{H}^S$-classes. Part (ii) holds by applying Proposition 9 and its dual. $\square$

We observe that neither condition (i) nor condition (ii) in the above proposition is sufficient on its own to provide a useful definition of index. If we were only to take (i) then every subgroup of a group would turn out to have finite index. To see that (ii) on its own is not strong enough note that when S is a combinatorial semigroup (that is, one all of whose maximal subgroups are trivial) condition (ii) is satisfied by any subsemigroup. This second observation also tells us that even strengthening (ii), by saying that there is a uniform bound on the number of $\mathcal{H}^T$-classes that any $\mathcal{H}^S$-class is partitioned into, is not strong enough.

For a subsemigroup T of a semigroup S, denote by $[S:T]_{\mathcal{R}}$ the number of (T-relative) $\mathcal{R}$-classes in $S \setminus T$.

Lemma 11. *If $Q \leq T \leq S$ then*

$$[S:T]_{\mathcal{R}} + [T:Q]_{\mathcal{R}} \leq [S:Q]_{\mathcal{R}} \leq [T:Q]_{\mathcal{R}} + [S:T]_{\mathcal{R}}([T:Q]_{\mathcal{R}}+1)^2.$$

Proof. Since $Q \leq T$ every $\mathcal{R}^Q$-class of S is either a subset of T or a subset of the complement of T. The number of $\mathcal{R}^Q$-classes in $T \setminus Q$ is precisely $[T:Q]_{\mathcal{R}}$. The $\mathcal{R}^Q$-classes in $S \setminus T$ give a finer partition than the $\mathcal{R}^T$-classes of $S \setminus T$. These observations prove that the left hand inequality holds.

Now we turn to the right hand inequality. Let R^T be an arbitrary $\mathcal{R}^T$-class of $S \setminus T$. We will prove that the number of $\mathcal{R}^Q$-classes in R^T is at most $([T:Q]_{\mathcal{R}}+1)^2$.

There is a natural ordering on the $\mathcal{R}^Q$-classes of R^T given by

$$R^Q_{s_1} \leq R^Q_{s_2} \Leftrightarrow s_1Q^1 \subseteq s_2Q^1.$$

Let P be the poset of $\mathcal{R}^Q$-classes of R^T under this ordering.

Claim 1. *Every chain in P has at most $[T:Q]_{\mathcal{R}}+1$ elements.*

Proof. Suppose, for the sake of a contradiction, that P has a chain of length k where $k > [T:Q]_{\mathcal{R}}+1$. It follows that there is a subset $\{s_1, s_2, \ldots, s_k\}$ of R^T such that:

$$s_1Q^1 \subsetneq s_2Q^1 \subsetneq \ldots \subsetneq s_kQ^1.$$

Therefore, for every $1 \leq i \leq k-1$ there is an element $q_i \in Q$ such that $s_{i+1}q_i = s_i$. For each $1 \leq i \leq k-1$ let $t_i \in T$ satisfy $s_it_i = s_{i+1}$; such elements exist since $s_1, \ldots, s_k$ are all $\mathcal{R}^T$-related. From $s_1t_1t_2\ldots t_i = s_{i+1}$ and $s_{i+1}Q \neq s_1Q$ it follows that $t_1t_2\ldots t_i \in T \setminus Q$ for all $i = 1, \ldots, k-1$. Since $k-1 > [T:Q]_{\mathcal{R}}$, two of these products have to belong to the same $\mathcal{R}^Q$-class, i.e.

$$t_1 \ldots t_i \mathcal{R}^Q t_1 \ldots t_i \ldots t_j$$

for some $1 \leq i < j \leq k-1$. But this implies

$$s_{j+1} = s_1t_1\ldots t_i \ldots t_j \mathcal{R}^Q s_1t_1\ldots t_i = s_{i+1},$$

a contradiction. □

Claim 2. *Every antichain in P has at most $[T:Q]_{\mathcal{R}}+1$ elements.*

Proof. Suppose, for the sake of a contradiction, that the poset P has an antichain with k elements where $k > [T : Q]_{\mathcal{R}}+1$. It follows that there is a subset $\{s_1, \ldots, s_k\}$ of R^T such that distinct elements in this set are not $\mathcal{R}^Q$-related, and such that for each $1 < j \leq k$ there exist $\alpha_j, \beta_j \in T \setminus Q$ such that:

$$s_1\alpha_j = s_j, \quad s_j\beta_j = s_1.$$

Since $\alpha_j \in T \setminus Q$ we can write

$$\alpha_j = r_j\gamma_j, \quad r_j = \alpha_j\delta_j$$

where $\gamma_j, \delta_j \in Q^1$, and the r_j come from a fixed set (of size $[T : Q]_{\mathcal{R}}$) of $\mathcal{R}^Q$-class representatives of $T \setminus Q$. Now we have

$$(s_1r_j)\gamma_j = s_1\alpha_j = s_j, \qquad s_j\delta_j = s_1\alpha_j\delta_j = s_1r_j.$$

It follows that $s_j\mathcal{R}^Q s_1r_j$. Since $k - 1 > [T : Q]_{\mathcal{R}}$ it follows that at least two of the r_j come from the same $\mathcal{R}^Q$-class, i.e. there are $1 < j_1 < j_2 \leq k$ with $r_{j_1} = r_{j_2}$. It then follows that

$$s_{j_1}\mathcal{R}^Q s_1r_{j_1} = s_1r_{j_2}\mathcal{R}^Q s_{j_2}$$

with $j_1 \neq j_2$ which is a contradiction. □

Returning to the proof of the lemma, by the above claims it follows that $|P| \leq ([T : Q]_{\mathcal{R}} + 1)^2$. Therefore, since R^T was arbitrary, the number of $\mathcal{R}^Q$-classes of $S \setminus T$ is at most $[S : T]_{\mathcal{R}}([T : Q]_{\mathcal{R}} + 1)^2$, and the second inequality holds. □

Corollary 12. *Let S be a semigroup and let $Q \leq T \leq S$. Then $[S : Q]_{\mathcal{R}}$ is finite if and only if $[S : T]_{\mathcal{R}}$ and $[T : Q]_{\mathcal{R}}$ are both finite.*

Of course, there are dual results concerning the $\mathcal{L}$-classes. Combining the two we obtain:

Corollary 13. *If S is a semigroup, T a subsemigroup of S, and Q a subsemigroup of T, then Q has finite Green index in S if and only if Q has finite index in T, and T has finite index in S.*

2.6 Intersections of subsemigroups

It is known that in a group the intersection of any two finite index subgroups is again a subgroup of finite index. Also it is obvious that in an infinite semigroup the intersection of two finite Rees index subsemigroups is non-empty and also has finite Rees index. In this subsection we provide an example showing that for finite Green index subsemigroups this is not true.

Example 14. Let $S = M^0[G; I, \Lambda; P]$ be a Rees matrix semigroup where G is any infinite group, $I = \Lambda = \{1, 2, \ldots, 10\}$, and $P = (p_{\lambda i})$ is given by

$$p_{\lambda i} = 1 \Leftrightarrow (\lambda - i \in \{0, 2\}) \text{ or } (\lambda = 9 \;\&\; i = 1) \text{ or } (\lambda = 10 \;\&\; i = 2).$$

(For the definition and more background on Rees matrix semigroups we refer the reader to [9].) Let T_1 be the subsemigroup generated by the set $E_1 = \{(i, g, i) : 1 \leq i \leq 10,\ g \in G\}$. Also let T_2 be the subsemigroup generated by the set $E_2 = \{(j + 2, g, j) : 1 \leq j \leq 10,\ g \in G\}$ with entries reduced mod 10 in the obvious way. Then it is straightforward to verify that T_1 and T_2 each have finite Green index in S, but $T_1 \cap T_2 = \{0\}$ which has infinite Green index in S.

2.7 Two easy finiteness conditions

Let us begin the main theme of this paper – the preservation of various finiteness conditions under subsemigroups and extensions of finite Green index – by considering the most basic finiteness condition: finiteness itself.

Theorem 15. *Let T be a subsemigroup of finite Green index in a semigroup S. Then S is finite if and only if T is finite, in which case $|S| \leq |T|[S : T]_G$.*

Proof. If S is finite then each of its subsemigroups is finite as well. Conversely, if T is finite, then every $\mathcal{H}$-class is finite, and, since $S \setminus T$ is a union of finitely many such classes, it follows that S is finite. Along with Proposition 5 (v), this also gives the desired inequality. □

We can equally easily prove:

Theorem 16. *Let T be a subsemigroup of finite Green index in a semigroup S. Then S has finitely many idempotents if and only if T has finitely many idempotents, in which case $|E(S)| \leq |E(T)| + [S : T]_G$.*

Proof. If S has finitely many idempotents then each of its subsemigroups has finitely many idempotents as well. The converse and the inequality follow from the fact that each of the finitely many $\mathcal{H}$-classes in $S \setminus T$ contains at most one idempotent (Proposition 4 (ii)). □

3 Local finiteness, periodicity, and right ideals

A semigroup S is called *locally finite* if every finitely generated subsemigroup of S is finite. A semigroup S is called periodic if for every $s \in S$ there exist $i, j > 0$, with $i \neq j$, such that $s^i = s^j$. Clearly, every locally finite semigroup is periodic, but the converse is not true. In the two results that follow we prove that both of these properties are inherited by finite Green index substructures, and by finite Green index extensions.

Theorem 17. *Let S be a semigroup and let T be a subsemigroup of S with finite Green index. Then S is locally finite if and only if T is locally finite (in which case all Schützenberger groups are locally finite as well).*

Proof. ($\Rightarrow$) Every subsemigroup of a locally finite semigroup is itself locally finite. It follows that T and all of the stabilizers $\mathrm{Stab}(H)$, where H is an $\mathcal{H}$-class in $S \setminus T$, are locally finite. Every homomorphic image of a locally finite semigroup is locally finite. It follows that, for each H, the Schützenberger group $\Gamma(H)$ is locally finite, since it is a homomorphic image of $\mathrm{Stab}(H)$.

($\Leftarrow$) Let $h_i \in H_i$ with $i \in I$ be fixed representatives of the $\mathcal{H}$-classes in $S \setminus T$, and define $h_1 = 1$. Let A be a finite subset of S. We will prove:

Claim 1. *There is a finite subset $\Sigma \subseteq T$ such that every $x \in \langle A \rangle$ can be written as $x = h_i t$ for some $\mathcal{H}$-class representative h_i, and some $t \in \langle \Sigma \rangle$.*

Since T is locally finite it will follow that $\langle \Sigma \rangle$ is finite and therefore, since I is finite, we conclude that $\langle A \rangle$ is finite.

To prove the claim, for each $a \in A$ and $i \in I \cup \{1\}$ we define $\rho(a, i) \in I \cup \{1\}$ as:

$$\rho(a, i) = \begin{cases} j & \text{if } ah_i \in H_j \\ 1 & \text{if } ah_i \in T. \end{cases}$$

We also define $\sigma(a, i) \in T$ such that

$$ah_i = h_{\rho(a,i)} \sigma(a, i)$$

where $\rho(a, i) \in I \cup \{1\}$ and $\sigma(a, i) \in T^1$. Now define

$$\Sigma = \{\sigma(a, i) : a \in A, i \in I \cup \{1\}\},$$

a finite subset of T^1. Let $x \in \langle A \rangle$ be arbitrary. Write $x = a_1 \dots a_k$ where $a_j \in A$ for all j. We will prove the claim by induction on k. When $k = 1$ we have

$$a_1 = a_1 \cdot h_1 = h_{\rho(a_1,1)} \sigma(a_1, 1)$$

which has the required form. Now suppose that $k > 1$ and that the result holds for all strictly smaller values of k; then we have

$$\begin{aligned} x &= a_1(a_2 \dots a_k) = a_1 h_j z \quad (\text{where } j \in I \ \& \ z \in \langle \Sigma \rangle) \\ &= h_{\rho(a_1,j)}\sigma(a_1,j)z \in h_{\rho(a_1,j)}\langle \Sigma \rangle \end{aligned}$$

completing the proof of the claim and of the theorem. □

Theorem 18. *Let S be a semigroup and let T be a subsemigroup of S with finite Green index. Then S is periodic if and only if T is periodic (in which case all Schützenberger groups are periodic as well).*

Proof. ($\Rightarrow$) Like local finiteness, periodicity is preserved by taking subsemigroups and homomorphic images, and the proof of this direction proceeds exactly as in Theorem 17.

($\Leftarrow$) Let $s \in S$ be an arbitrary element. If $s^i \in T$ for some i then $s^{ij} = s^{ik}$ for some $j \neq k$, since T is periodic. Otherwise $s^i \in S \setminus T$ for all $i \geq 1$. Since T has finite index in S it follows that $s^m \mathcal{H} s^{m+r}$ for some m and r. Now since $\mathcal{R}$ is a left congruence, $s^m \mathcal{R} s^{m+r}$ implies that $s^{m+r} \mathcal{R} s^{m+2r}$. Similarly since $\mathcal{L}$ is a right congruence, $s^m \mathcal{L} s^{m+r}$ implies that $s^{m+r} \mathcal{L} s^{m+2r}$. Therefore $s^m \mathcal{H} s^{m+2r}$. Repeating the argument we conclude that $s^m \mathcal{H} s^{m+qr}$ for all $q \in \mathbb{N}$. Choose z so that $0 \leq z \leq r-1$ and $m+z \equiv 0 \pmod{r}$. Then, with $m+z = kr$, we have:

$$(s^{m+z})^2 = s^{m+(m+z)}s^z = s^{m+kr}s^z \mathcal{H} s^m s^z = s^{m+z}.$$

It follows from Proposition 4 (ii) that $H_{a^{m+z}}$ is a group $\mathcal{H}$-class of $S \setminus T$; by Proposition 5 (vii) this group is isomorphic to the Schützenberger group $\Gamma(H)$, which, in turn, is periodic since it is the homomorphic image of a subsemigroup of T. It follows that $s^{(m+z)l} = s^{(m+z)p}$ for some $p \neq l$. Therefore S is periodic. □

We now consider the finiteness condition of having finitely many right ideals. The following result generalizes [22, Theorem 10.4]. Clearly a semigroup S has finitely many right ideals if and only if it has finitely many $\mathcal{R}^S$-classes.

Theorem 19. *Let S be a semigroup and let T be a subsemigroup of S with finite Green index. Then S has finitely many right ideals if and only if T has finitely many right ideals.*

Proof. ($\Leftarrow$) If T has finitely many $\mathcal{R}^T$-classes then, since T has finite Green index in S, it follows that S has finitely many $\mathcal{R}^T$-classes. The result now follows from the fact that each $\mathcal{R}^S$-class of S is a disjoint union of $\mathcal{R}^T$-classes.

($\Rightarrow$) Suppose that S has finitely many $\mathcal{R}^S$-classes. Then by Proposition 9 each of these $\mathcal{R}^S$-classes is a union of finitely many $\mathcal{R}^T$-classes. Therefore S is a union of finitely many $\mathcal{R}^T$-classes. In particular T is a union of finitely many $\mathcal{R}^T$-classes. $\square$

Of course one may ask the same question for two-sided ideals. If T has finitely many two-sided ideals then by an argument very similar to that used in the proof of Theorem 19 it follows that S has only finitely many two-sided ideals. The converse, if true, could be a difficult problem since it would both answer and generalize [22, Open Problem 11.3(i)].

4 Residual finiteness

Let X be a set and let π be an equivalence relation on X. For $x \in X$ we use x/π to denote the equivalence class of the element x, and X/π to denote the set of all equivalence classes. We let $[X : \pi]$ denote the number of π-classes of X and call this the *index* of the relation π. We say that π *separates* the elements s and t if $s/\pi \neq t/\pi$. Given a set X we use Φ_X to denote the full relation $X \times X$, and Δ_X for the diagonal $\{(x, x) : x \in X\}$.

A semigroup S is *residually finite* if for every two distinct $x, y \in S$ there is a congruence σ on S which has finite index and which separates x and y. This is equivalent to saying that there is a homomorphism ϕ from S onto a finite semigroup with the property that $x\phi \neq y\phi$. The property of a semigroup being residually finite is equivalent to that of being a subdirect product of finite semigroups. Residual finiteness is also closely connected with algorithmic problems; for example, any finitely presented residually finite semigroup has solvable word problem [5].

The purpose of this section is to prove the final remaining statement of Theorem 3 from Section 1:

Theorem 20. *Let S be a semigroup and let T be a subsemigroup of S with finite Green index. Let $\{H_i : i \in I\}$ be the set of $\mathcal{H}$-classes of the complement $S \setminus T$. Then S is residually finite if and only if T and all the Schützenberger groups $\Gamma(H_i)$ are residually finite.*

Central to the proof will be the concept of action and its relationship with congruences, and we begin by reviewing the basic notions we need. For more background on semigroup actions we refer the reader to [4], [9].

Let S be a monoid with identity element 1_S, and let X be a set. A *right action* of S on X is a mapping $X \times S \to X, (x, s) \mapsto xs$, satisfying $x(st) = (xs)t$, and $x1_S = x$ for all $s, t \in S$, $x \in X$. If $s, t \in S$ satisfy $xs = xt$ for all $x \in X$ then we say that s and t *act in the same way* on X.

A monoid S acts on itself by right multiplication. More generally, if T is a submonoid of S then T acts on S by right multiplication. We say that an equivalence relation ρ on X is a *congruence* of the action of S on X if $x\rho y$ implies $xs \,\rho\, ys$ for all $x, y \in X$, $s \in S$. In this situation, there is a natural action of S on the set X/ρ of equivalence classes, given by $(x/\rho)s = (xs)/\rho$. In particular, if $Y \subseteq X$ is closed under the action of S (in the sense that $YS \subseteq Y$) then we can define X/Y as the quotient of X by the congruence $(Y \times Y) \cup \Delta_X$. A particular example of this is given by taking a (T-relative) $\mathcal{R}$-class R in S, along with the symbol $0 \notin S$, and then defining the action of T on $R \cup \{0\}$ by

$$r \cdot t = \begin{cases} rt \text{ if } rt \in R \\ 0 \quad \text{otherwise.} \end{cases}$$

Also, since $\mathcal{L}^T$ is a right congruence on S, the monoid S acts on the set of $\mathcal{L}$-classes of S via right multiplication.

Given an action of S on X and an element $x \in X$ the *orbit* of x is the set $\mathcal{O}(x) = \{xs : s \in S\}$, while the *strong orbit* of x is $\mathcal{SO}(x) = \{y \in \mathcal{O}(x) : x \in \mathcal{O}(y)\}$. If S happens to be a group then $\mathcal{O}(x) = \mathcal{SO}(x)$. Also, in this terminology, the (T-relative) $\mathcal{R}$-classes in S are simply the strong orbits of the natural action of T^1 on S via right multiplication.

Following [16] we may associate a group with every subset $H \subseteq X$ in the following way. First we define

$$\mathrm{Stab}(H) = \{s \in S : x \mapsto xs \text{ is a bijection from } H \text{ onto } H\},$$

and then let σ be a relation on $\mathrm{Stab}(H)$ where $(s, t) \in \sigma$ if and only if $xs = xt$ for all $x \in H$. The quotient $\Gamma(H) = \mathrm{Stab}(H)/\sigma$ is a group called the *generalized Schützenberger group of* H. In particular, if we consider the natural action of T^1 on S, and we let H be some (T-relative) $\mathcal{H}$-class of S then the generalized Schützenberger group of H is exactly the same as the Schützenberger group as defined in Section 2.

We say that an S-act X is *residually finite* if for any $x, y \in X$ with $x \neq y$ there is a congruence ρ on X with finitely many ρ-classes such that $x\rho \neq y\rho$. The following result is required for the proof of the direct part of Theorem 20.

Proposition 21. *If X is a residually finite S-act, then every generalized Schützenberger group $\Gamma(H)$ $(H \subseteq X)$ is residually finite.*

Proof. Let $s/\sigma, t/\sigma \in \Gamma(H)$ with $s/\sigma \neq t/\sigma$. We want to find a finite index congruence on the group $\Gamma(H)$ separating s/σ and t/σ. Since $s/\sigma \neq t/\sigma$ it follows that for some $x \in H$ we have $xs \neq xt$. Since the act X is residually finite there is a finite index congruence ρ on X such that $(xs)/\rho \neq (xt)/\rho$. The restriction of the equivalence relation ρ to H is an equivalence relation on H. Let H/ρ denote the set of equivalence classes of this restriction, noting that H/ρ is finite and that xs and xt belong to different ρ-classes of H.

We claim that the group $\Gamma(H) = \mathrm{Stab}(H)/\sigma$ acts on the finite set H/ρ by the following rule

$$(h/\rho) \cdot (u/\sigma) = (hu)/\rho$$

where $h \in H$ and $u \in \mathrm{Stab}(H)$. We just have to check that this action is well-defined.

Let $h, h' \in H$ with $(h, h') \in \rho$, and let $u, u' \in \mathrm{Stab}(H)$ with $(u, u') \in \sigma$. We must show that $(hu, h'u') \in \rho$. Indeed, since $(h, h') \in \rho$, and ρ is a right congruence, it follows that $(hu, h'u) \in \rho$. But since $(u, u') \in \sigma$ it follows from the defintion of σ that $h'u = h'u'$. We conclude that $(hu, h'u') \in \rho$ as required.

Therefore the group $\Gamma(H)$ acts on the finite set H/ρ by the above rule. Now define a congruence σ' on $\Gamma(H)$ by

$$(s'/\sigma)\ \sigma'\ (t'/\sigma) \Leftrightarrow (\forall z \in H/\rho)\ z \cdot (s'/\sigma) = z \cdot (t'/\sigma).$$

Since H/ρ is finite the congruence σ' has finite index in $\Gamma(H)$. Also, it separates s/σ and t/σ since by the choice of $x \in H$ above we have

$$(x/\rho) \cdot (s/\sigma) = (xs)/\rho \neq (xt)/\rho = (x/\rho) \cdot (t/\sigma).$$

□

Next we make the following observation:

Proposition 22. *If S is a residually finite semigroup and T a subsemigroup of S, then S considered as a right T-act is also residually finite.*

Proof. Any congruence of (the semigroup) S is also a congruence of the T-act S. □

Combining Propositions 21, 22 and the obvious fact that a subsemigroup of a residually finite semigroup is residually finite we obtain:

Proposition 23. *If S is a residually finite semigroup and T a subsemigroup of S, then T and all (T-relative) Schützenberger groups $\Gamma(H)$ are residually finite.*

This is (stronger than) the direct part of Theorem 20. The remainder of this section will be devoted to proving the converse direction. The main task will be to use congruences of finite index on the subsemigroup T and Schützenberger groups $\Gamma(H)$ to define appropriate congruences on S, while maintaining finite index and certain other properties. In what follows we shall build up a store of technical results concerning such constructions.

4.2 Refining one-sided congruences

Every equivalence relation π on the semigroup S gives rise to a right congruence $\Sigma_r(\pi)$ which is maximal amongst right congruences ρ of S that satisfy $\rho \subseteq \pi \subseteq S \times S$. This right congruence is given by:

$$\Sigma_r(\pi) = \{(x, y) \in S \times S : (xs, ys) \in \pi \text{ for all } s \in S^1\}.$$

The proof of this is a straightforward modification of the analogous two-sided statement [9, Proposition 1.5.10].

This can be expressed in terms of equivalence classes as follows. For $s \in S$ and $X \subseteq S$ we define

$$Q_S(s, X) = \{t \in S^1 : st \in X\}.$$

Proposition 24. *[24, Proposition 2.2] Let S be a semigroup, let π be any equivalence relation on S, and let C_i ($i \in I$) be all the equivalence classes of π in S. Then for arbitrary $x, y \in S$ we have*

$$(x, y) \in \Sigma_r(\pi) \Leftrightarrow Q_S(x, C_i) = Q_S(y, C_i) \quad \textit{for all } i \in I.$$

Similarly we can define $\Sigma_l(\pi)$, the largest left congruence of S contained in π, and $\Sigma(\pi)$, the largest two-sided congruence of S contained in π.

The next lemma tells us that any finite index right congruence on S can be refined to a finite index two-sided congruence. As a consequence, in order to show that a semigroup S is residually finite it is sufficient to prove that, given $x, y \in S$, there is a right congruence ρ, with finite index, such that $x/\rho \neq y/\rho$.

Lemma 25. *[24, Theorem 2.4] Let ρ be a right congruence on S. If ρ has finite index then $\Sigma(\rho)$ has finite index.*

Let $T \leq S$ be a subsemigroup with finite Green index and let $H \subseteq S \setminus T$ be an $\mathcal{H}$-class with $h \in H$. Also let R be the $\mathcal{R}$-class of S containing H. Let $N \trianglelefteq \Gamma(H)$ be a normal subgroup with finite index. Let N_i with $0 \leq i \leq m$ be the cosets of N in $\Gamma(H)$ where $N_0 = N$. For each $i = 0, \ldots, m$ define

$$\overline{N_i} = \{t \in \mathrm{Stab}(H) : t/\sigma \in N_i\}.$$

Now partition H as $H = \bigcup_{0 \leq i \leq m} C_i$ where $C_i = h\overline{N_i}$. Observe that the C_i blocks of H are preserved under right multiplication from $\mathrm{Stab}(H)$. Indeed, for $x, y \in C_i$, $t \in \mathrm{Stab}(H)$ we have

$$xt \in C_j \Leftrightarrow t/\sigma \in N_i^{-1}N_j \Leftrightarrow yt \in C_j.$$

As described above, T^1 acts on $R \cup \{0\}$ ($0 \notin S$) by right multiplication. Let H_j with $0 \leq j \leq p$ be the set of $\mathcal{H}$-classes of R, where $H_0 = H$. For each $j = 0, \ldots, p$, fix elements $t_j, t_j' \in T^1$ such that $H_0 t_j = H_j$, $H_j t_j' = H_0$, and so that for all $k \in H_0$, $k' \in H_j$ we have $kt_jt_j' = k$, $k't_j't_j = k'$. Such elements must exist as a consequence of Proposition 4 (iii); in addition, we stipulate $t_0 = t_0' = 1$. Define sets $C_{i,j}$ by $C_{i,j} = C_i t_j$ where $0 \leq i \leq m$ and $0 \leq j \leq p$. In particular we have $C_i = C_i t_0 = C_{i,0}$ for all $0 \leq i \leq m$. By Proposition 4 (iii), the $\mathcal{R}$-class R is equal to the disjoint union of the sets $C_{i,j}$ ($0 \leq i \leq m$, $0 \leq j \leq p$). Moreover, the blocks of this partition are preserved by the action of T, in the sense that for every $C_{i,j}$, and every $t \in T$, either $C_{i,j}t \cap R = \emptyset$ or $C_{i,j}t \subseteq R$ and we have

$$C_{i,j}t = C_{i,j}tt_f't_f = C_i(t_jtt_f')t_f = C_kt_f = C_{k,f}$$

where $H_jt = H_f$, and $C_i(t_jtt_f{}') = C_k$ with $t_jtt_f{}' \in \mathrm{Stab}(H)$.

The following lemma shows how any finite index congruence on T can be refined to give a finite index congruence which has the property that congruent elements act in the same way on the $\mathcal{L}$-classes of $S \setminus T$, and also act in the same way on the blocks of the partition $R = \bigcup_{0 \leq i \leq m} \bigcup_{0 \leq j \leq p} C_{i,j}$ of R.

Lemma 26. *Given a finite index congruence ρ on T there exists a finite index congruence ρ' on T with the following properties:*

(1) ρ' refines ρ (i.e. $\rho' \subseteq \rho$);
(2) if $x\rho'y$ then x and y act in the same way on the $\mathcal{L}$-classes of $S \setminus T$;
(3) if $x\rho'y$ then x and y act on the blocks $C_{i,j}$ in the same way.

Proof. The action of T on the finite set of blocks $C_{i,j}$ (together with the extra symbol 0) is an action on a finite set, and thus gives rise to a finite index congruence ρ_C on T where two elements are related if and only if they act in the same way.

Since $\mathcal{L}$ is a right congruence, T acts on the set of $\mathcal{L}$-classes of $S \setminus T$ (together with the extra symbol 0) via right multiplication. There are only finitely many $\mathcal{L}$-classes, so this action induces a finite index congruence ρ_L on T where two elements are related if they act on the $\mathcal{L}$-classes of $S \setminus T$ in the same way.

The congruence $\rho' = \rho \cap \rho_C \cap \rho_L$ has finite index and all the properties of the statement of the lemma. $\square$

4.4 Extending a congruence from T to S.

Let all the notation be as in the preceding subsection. Furthermore, let L_i with $0 \leq i \leq n$ be the set of $\mathcal{L}$-classes of $S \setminus T$, where $H_0 \subseteq L_0$, and let ${L_0}' = L_0 \setminus H_0$. Now partition S as

$$S = T \cup L_1 \cup L_2 \cup \ldots \cup L_n \cup {L_0}' \cup C_0 \cup C_1 \cup \ldots \cup C_m.$$

Denote the equivalence relation corresponding to this partition by $\pi(H, N)$. Note that $\pi(H, N)$ depends only on H and N and not on the choice of representative $h \in H$.

The next lemma is fundamental. It contains most of the technical details that are required for the proof of the main theorem of this section.

Lemma 27. *If ρ is a finite index right congruence on T and $\pi(\rho, H, N) = \rho \cap \pi(H, N)$, then the right congruence $\Sigma_r(\pi(\rho, H, N))$ on S has finite index.*

Proof. First of all we may assume, without loss of generality, that ρ has the properties listed in Lemma 26: ρ-related elements act in the same way on $\mathcal{L}$-classes, and on $C_{i,j}$ blocks of R. Let D_j with $1 \leq j \leq q$ be the ρ-classes of T, so that the equivalence relation $\pi(\rho, H, N)$ partitions the semigroup S as:

$$\begin{aligned} S &= T \cup (U \setminus H) \cup H \\ &= D_1 \cup D_2 \cup \ldots \cup D_q \cup L_1 \cup L_2 \cup \ldots \cup L_n \cup {L_0}' \cup C_0 \cup C_1 \cup \ldots \cup C_m, \end{aligned}$$

where U stands for $S \setminus T$. It follows from Proposition 24 that proving that $\Sigma_r(\pi(\rho, H, N))$ has finite index is equivalent to proving that there are only finitely many possible sets of the form $Q_S(x, \mathcal{C})$ where $x \in S$ and $\mathcal{C}$ is a block of the above partition (i.e. D_i, L_j, C_k or L_0'). This gives four cases that we must consider. Choose and fix a set of representatives for the set of $\mathcal{R}^T$- and set of $\mathcal{L}^T$-classes, respectively, contained in U. Of course, both these sets of representatives are finite.

Case 1. $\mathcal{C} = C_i = C_{i,0}$ for some $0 \le i \le m$.

Note that $Q_S(x, C_i) = Q_U(x, C_i) \cup Q_T(x, C_i)$. We will show that (as x varies) $Q_U(x, C_i)$ and $Q_T(x, C_i)$ each take only finitely many distinct values, and then the same will follow for $Q_S(x, C_i)$.

First consider $Q_U(x, C_i)$. Let $u \in Q_U(x, C_i)$ be arbitrary. Let r be the representative of the $\mathcal{R}$-class of u. Since $\mathcal{R}$ is a left congruence and $r\mathcal{R}u$ it follows that $xr\mathcal{R}xu$. But $xu \in C_i \subseteq R$, the $\mathcal{R}$-class of H, and so $xr \in C_{k,l}$ for some k and l. Since $r\mathcal{R}u$ we can write $u = rt_1$, where $t_1 \in T^1$. Now $xu = xrt_1$ and it follows that

$$\{t_1' \in T^1 : xrt_1' \in C_i\} = \{t_1' \in T^1 : C_{k,l}t_1' = C_{i,0}\},$$

and this set is a union of ρ-classes, since ρ-related elements act in the same way on the $C_{c,d}$ blocks. It follows that for each $x \in S$, $Q_U(x, C_i)$ is equal to a union of sets of the form rZ where r is a representative of an $\mathcal{R}$-class in U, and Z is a union of ρ-classes of T. Since U has finitely many $\mathcal{R}$-classes, and ρ has finite index, it follows that there are only finitely many possibilities for the set $Q_U(x, C_i)$.

Now consider the set $Q_T(x, C_i)$. If $x \in T$ then $Q_T(x, C_i)$ is empty because $T \le S$, so suppose $x \in U$. Let $t \in Q_T(x, C_i)$ be arbitrary. Let l be the $\mathcal{L}$-class representative of x. Since $\mathcal{L}$ is a right congruence, $x\mathcal{L}l$ implies that $xt\mathcal{L}lt$. Since $xt \in H \subseteq L_0$ it follows that $lt \in H' \subseteq L_0$ where H' is some $\mathcal{H}$-class of S in U. Since H and H' are in the same $\mathcal{L}$-class it follows from Proposition 5 that $\mathrm{Stab}(H') = \mathrm{Stab}(H)$ and we can partition H' into blocks $C_i' = h'\overline{N}_i$, where h' is a fixed element of H'. These blocks have the property that for all j, and all $t'' \in T$, we have $C_j't'' = C_j'$ if and only if $t'' \in \overline{N}$. Suppose that $lt \in C_j'$. Now we claim that

$$\{t' \in T : xt' \in C_i\} = \{t' \in T : lt' \in C_j'\}. \tag{1}$$

Indeed, if $xt' \in C_i$ then since $xt \in C_i$ we can write $xt' = xtt_2$ for some $t_2 \in T^1$. Now $xt \in C_i$ and $(xt)t_2 \in C_i$ which implies that $C_it_2 = C_i$. It follows by the comment above that $t_2 \in \overline{N}$ and therefore that $C_j't_2 = C_j'$. Since $l\mathcal{L}x$ we can write $l = t_3x$ where $t_3 \in T$. Now we have

$$lt' = t_3xt' = t_3xtt_2 = (lt)t_2 \in C_j't_2 = C_j'$$

proving the direct inclusion in (1). For the converse inclusion, suppose $lt' \in C_j'$. Since $lt \in C_j'$ we can write $lt' = ltt_4$ for some $t_4 \in T^1$. Now $lt \in C_j'$ and $(lt)t_4 \in C_j'$ which implies $C_j't_4 = C_j'$. It follows, as before, that $C_jt_4 = C_j$. Since $x\mathcal{L}l$ we can write $x = t_5l$ where $t_5 \in T$. Now we have:

$$xt' = t_5lt' = t_5ltt_4 = (xt)t_4 \in C_it_4 = C_i,$$

as required. There are only finitely many possible sets $\{t' \in T : lt' \in C_j'\}$, since

U has only finitely many $\mathcal{L}$-classes, and each of the finitely many $\mathcal{H}$-classes of L_0 has only finitely many C'_j blocks. This completes the proof of Case 1.

Case 2. $\mathcal{C} = D_i$ for some $1 \leq i \leq q$.

First we consider the case where $x \in U$. Let l be the representative for the $\mathcal{L}$-class of x. We claim that if $Q_S(x, D_i)$ is non-empty, there exists $j \in \{1, \ldots, q\}$ such that $Q_S(x, D_i) = Q_S(l, D_j)$. Write $l = t_1 x$, $x = t_2 l$, where $t_1, t_2 \in T^1$. Since ρ is a congruence on T, it follows that $t_1 D_i \subseteq D_j$ for some $j \in \{1, \ldots, q\}$. Let $s \in Q_S(x, D_i)$ so that $xs \in D_i$, and $t_1(xs) \in D_j$. Then:

$$t_2(t_1 xs) = (t_2 l)s = xs \in D_i.$$

Since $t_1 xs \in D_j$ and ρ is a congruence it follows that $t_2 D_j \subseteq D_i$. Now we claim that

$$Q_S(l, D_j) = \{s \in S : ls \in D_j\} = \{s \in S : xs \in D_i\} = Q_S(x, D_i).$$

Indeed, if $s' \in Q_S(x, D_i)$ so that $xs' \in D_i$, then $ls' = (t_1 x)s' = t_1(xs') \in t_1 D_i \subseteq D_j$ which implies that $s' \in Q_S(l, D_j)$. Conversely, if $s' \in Q_S(l, D_j)$ so that $xs' \in D_j$, then $xs' = (t_2 l)s' = t_2(ls') \in t_2 D_j \subseteq D_i$ which implies that $s' \in Q_S(x, D_i)$.

We conclude from this that since U has only finitely many $\mathcal{L}$-classes there are only finitely many possible sets $Q_S(x, D_i)$ where $x \in U$ and $1 \leq i \leq q$.

Now consider $x \in T$. Since ρ is a congruence it follows that $Q_T(x, D_i)$ is a union of ρ-classes, and so there are only finitely many possibilities for this set. Now consider $Q_U(x, D_i)$. Let $u \in Q_U(x, D_i)$ be arbitrary. Let r be the representative of the $\mathcal{R}$-class of u, and write $u = rt_3$ where $t_3 \in T^1$. Since $\mathcal{R}$ is a left congruence $u\mathcal{R}r$ implies $xu\mathcal{R}xr$ and so $xr \in T$. Suppose $xr \in D_j$. Then $xrt_3 = xu \in D_i$ and $\{t' \in T : (xr)t' \in D_i\} = \{t' \in T : D_j t' \subseteq D_i\}$, which is a union of ρ-classes of T. It follows that if $x \in T$ then $Q_U(x, D_i)$ is a union of sets of the form rZ where r is a representative of an $\mathcal{R}$-class in U, and Z is a union of ρ-classes of T; there are clearly only finitely many sets of this form.

Case 3. $\mathcal{C} = L_i$ for some $1 \leq i \leq n$.

If $x \in U$ then $Q_T(x, L_i)$ is a union of ρ-classes, since ρ-related elements act on the $\mathcal{L}$-classes of U in the same way. Let $u \in Q_U(x, L_i)$ be arbitrary. Let r be the representative of the $\mathcal{R}$-class of u. Since $\mathcal{R}$ is a left congruence $r\mathcal{R}u$ implies $xr\mathcal{R}xu$. Suppose that $xr \in L_j$, and write $u = rt_1$ where $t_1 \in T^1$. Now $xrt_1 = xu \in L_i$ so $L_j t_1 = L_i$, and we have $\{t' \in T : xrt' \in L_i\} = \{t' \in T : L_j t' = L_i\}$. Since ρ related elements act on the $\mathcal{L}$-classes of U in the same way, and since there are only finitely many $\mathcal{R}$-classes in $S \setminus T$, we conclude as before that there are only finitely many possibilities for the set $Q_U(x, L_i)$.

Now suppose that $x \in T$. The set $Q_T(x, L_i)$ is empty since $T \leq S$ and $L_i \subseteq U$. Let $u \in Q_U(x, L_i)$ be arbitrary. Let r be the representative of the $\mathcal{R}$-class of u. Since $\mathcal{R}$ is a left congruence, $r\mathcal{R}u$ implies $xr\mathcal{R}xu$. Let L_j be the $\mathcal{L}$-class to which xr belongs. Write $u = rt'$ so that $xu = xrt' \in L_i$. It follows that $L_j t' = L_i$ and that $\{t' \in T : xrt' \in L_i\} = \{t' \in T : L_j t' \subseteq L_i\}$. This set is a union of ρ-classes of T, since ρ related elements act on the $\mathcal{L}$-classes of U in the same way. So again, there are only finitely many possibilities for the set $Q_S(x, L_i)$.

Case 4. $\mathcal{C} = L'_0$.

Clearly, $s \in Q_S(x, L'_0)$ if and only if s does not belong to any of the sets $Q_S(x, \mathcal{C})$ where $\mathcal{C}$ is one of D_i, L_j or C_k. In Cases 1, 2, 3 we have proved that there are only finitely many sets $Q_S(x, \mathcal{C})$ for such $\mathcal{C}$, and therefore it follows that there are only finitely many sets of the form $Q_S(x, L'_0)$. $\square$

Now we can prove the main theorem of this section.

Proof of Theorem 20. ($\Rightarrow$) This is an immediate corollary of Proposition 23.

($\Leftarrow$) Suppose that T is residually finite and that every group $\Gamma(H_i)$ is residually finite. Let $x, y \in S$, $x \neq y$. By Lemma 25, to prove that S is residually finite, it is sufficient to find a right congruence ρ on S with finite index that separates x and y.

If $x, y \in T$, since T is residually finite there is a congruence ρ on T which has finite index and separates x and y. By Lemma 27 the right congruence $\Sigma_r(\pi(\rho, H, \Gamma(H)))$, where H is any $\mathcal{H}$-class of $S \setminus T$, has finite index and separates x and y.

If $x \in T$ and $y \in S \setminus T$ or if $x, y \in S \setminus T$ and x is not $\mathcal{L}$-related to y, then by Lemma 27 for any $\mathcal{H}$-class H in $S \setminus T$ the right congruence $\Sigma_r(\pi(\Phi_T, H, \Gamma(H)))$ on S has finite index and separates x and y. By left-right duality, there is a left congruence (and hence a two-sided congruence) separating $x, y \in S \setminus T$ which are not $\mathcal{R}$-related.

Finally, suppose that $x, y \in S \setminus T$ and that $x\mathcal{H}y$. It is only in this case that we are going to make use of the hypothesis that the Schützenberger groups are residually finite. Let H be the common $\mathcal{H}$-class of x and y, and fix an element $h \in H$. Let $t_x, t_y \in \mathrm{Stab}(H)$ satisfy $ht_x = x$ and $ht_y = y$. Let $\Gamma(H) = \mathrm{Stab}(H)/\sigma$ be the Schützenberger group of H. Since $\Gamma(H)$ is residually finite, there is a normal subgroup $N \trianglelefteq \Gamma(H)$ of finite index such that t_x/σ and t_y/σ belong to different cosets N_i, N_j of N in $\Gamma(H)$. By Lemma 27 the right congruence $\nu = \Sigma_r(\pi(\Phi_T, H, N))$ respects both $C_i = h\overline{N_i}$ and $C_j = h\overline{N_j}$, and, since $x \in C_i$, $y \in C_j$, we conclude that ν separates x and y. $\square$

The theorem may now be applied to recover the corresponding results for subgroups with finite subgroup index, and subsemigroups of semigroups with finite Rees index.

Corollary 28. *For groups the property of being residually finite is inherited by finite index subgroups and finite index extensions.*

Proof. Let G be a group and let K be a subgroup with finite index. In particular, K has finite Green index in G. If G is residually finite then by Proposition 23 so is K. Conversely suppose that K is residually finite. Let H be an arbitrary $\mathcal{H}^K$ class of the complement $G \setminus K$. Since K is a group it follows that $\mathrm{Stab}(H)$ is a subgroup of K. From the definition of the congruence $\sigma = \sigma(H)$, since G is group it follows that σ is the diagonal relation and $\Gamma(H) = \mathrm{Stab}(H)/\sigma = \mathrm{Stab}(H)$. So in this case $\Gamma(H)$ is actually isomorphic to a subgroup of K. Since K is residually finite and $\Gamma(H)$ is isomorphic to a subgroup of K it follows that $\Gamma(H)$ is residually finite. Since H was an arbitrary $\mathcal{H}^K$-class it follows that all of the Schützenberger groups are residually finite. Now by Theorem 20 since K and all the Schützenberger groups are residually finite, and K has finite Green index in G, it follows that G is residually finite. □

Corollary 29. *For semigroups the property of being residually finite is inherited by finite Rees index subsemigroups and finite Rees index extensions.*

Proof. Let S be a semigroup and let T be a subsemigroup with finite Rees index. In particular, T has finite Green index in S. If S is residually finite then so is T, by Proposition 23. Conversely, suppose that T is residually finite. Since the complement is finite it follows that every Schützenberger group $\Gamma(H)$ is finite and therefore is residually finite. Now by Theorem 20 since T and all the Schützenberger groups are residually finite, and T has finite Green index in S, it follows that S is residually finite. □

If we remove the condition that the Schützenberger groups are residually finite then Theorem 20 does not hold, as the following example demonstrates.

Example 30. Let G be the free group of rank 2 and let $\phi : G \to H$ be a homomorphism from G onto a non-residually finite group H. Let $S = G \cup H$ with multiplication defined in the following way. Given $x, y \in S$ if $x, y \in G$ then we multiply as in G; if $x, y \in H$ then multiply as in H; if $x \in G$ and $y \in H$ take the product of $\phi(x)$ and y in H; if $x \in H$ and $y \in G$ take the product of x and $\phi(y)$ in H. This is an example of a so called Clifford monoid (see [9, Chapter 4, Section 4.2]). In this example G is residually finite and has Green index 2 in S, but S is not residually finite since H is not residually finite.

5 Green index and syntactic index

In [24] the concept of syntactic index was introduced. Let S be a semigroup and let T be a subsemigroup of S. The *right syntactic index* $[S:T]_{rs}$ of T in S is the number $[S:\Sigma_r(\Phi_T \cup \Phi_{S\setminus T})]$ of equivalence classes of the largest right congruence on S which respects the partition $S = T \cup (S \setminus T)$. The *left syntactic index* $[S:T]_{ls}$ is defined analogously. In [24] it is proved that $[S:T]_{rs}$ is finite if and only if $[S:T]_{ls}$ is finite. If T satisfies either of these two equivalent conditions we say that T has finite syntactic index in S.

We now establish the relationship between Green index and syntactic index. To see that finite syntactic index does not imply finite Green index consider the semigroup $S = Y \times G$ where G is an infinite group and Y is the two element semilattice $\{0,1\}$ with multiplication $x \cdot y = \min(x,y)$. (In fact, S is another example of a Clifford monoid.) If $T = \{(0,g) : g \in G\}$ then T has finite syntactic index in S since $\Phi_T \cup \Phi_{S\setminus T}$ is a congruence. However, T has infinite Green index in S since T is an ideal with infinite complement. On the other hand, the converse does hold as we now demonstrate.

Theorem 31. *Let S be a semigroup and let T be a subsemigroup of S. If the Green index of T in S is equal to $i \in \mathbb{N}$ then $[S:T]_{rs} \leq 2^i + i$. In particular, if the Green index of T in S is finite then the syntactic index is finite.*

Proof. We let $U = S \setminus T$, and prove that $\rho = \Sigma_r(\Phi_T \cup \Phi_U)$ has no more than $2^i + i$ congruence classes. By Proposition 24 for $x, y \in S$

$$(x,y) \in \rho \Leftrightarrow Q_S(x,U) = Q_S(y,U) \Leftrightarrow Q_S(x,T) = Q_S(y,T).$$

First consider the ρ-classes in T. Let $x \in T$ and consider $Q_S(x,T)$. Since $T \leq S$, and $x \in T$ it follows that $Q_T(x,T) = T$. Also since $\mathcal{R}$ is a left congruence which respects the partition $S = T \cup U$, it follows that for all $u, v \in U$ with $u\mathcal{R}v$, we have $u \in Q_U(x,T)$ if and only if $v \in Q_U(x,T)$. It follows that $Q_S(x,T)$ is equal to $T \cup V$ where V is a union of $\mathcal{R}$-classes of U. The number of $\mathcal{R}$-classes in U is at most i, and so there can be at most 2^i different sets $Q_U(x,T)$. Therefore the number of ρ-classes in T is at most 2^i.

Now consider the ρ-classes in U. Since $\mathcal{L}$ is a right congruence which respects the partition $S = T \cup U$, it follows that if $u\mathcal{L}v$ for some $u, v \in U$ then $us\mathcal{L}vs$ for all $s \in S$, which implies that $Q_S(u,T) = Q_S(v,T)$, which then implies that $(u,v) \in \rho$. Hence the number of ρ-classes in U is no more than the number of $\mathcal{L}$-classes in U, which in turn is no more than i. Combining our upper bounds for the numbers of ρ-classes in T and U respectively yields the result. □

The above link between the Green and syntactic indices has the following important consequence.

Corollary 32. *If S is a finitely generated semigroup then S has only finitely many subsemigroups of any given finite Green index n.*

Proof. If $T \leq S$ has Green index n, then $[S : T]_{rs} \leq 2^n + n$ by Theorem 31. But S has only finitely many subsemigroups of any finite syntactic index by [24, Theorem 3.2 (iv)], and the assertion follows. $\square$

We can also show that our new approach will not shed new light on questions concerning (non-group) subsemigroups of groups.

Proposition 33. *Let S be a group and let T be a subsemigroup of S. If T has finite syntactic index then T is a subgroup of S (with finite index).*

Proof. Since T has finite syntactic index it follows by definition and Lemma 25 that there is a congruence σ on S with finitely many classes such that T is a union of σ-classes. Since S is a group it follows that S has a normal subgroup N with finite subgroup index such that T is a union of cosets of N. Since S/N is finite and every subsemigroup of a finite group is in fact a subgroup, it follows that T is a subgroup of S. $\square$

Combining this with Theorem 31 we obtain the following.

Corollary 34. *Let S be a group and let T be a subsemigroup of S. If T has finite Green index then T is a subgroup of S (with finite index).*

6 Applications, examples and remarks

We have already observed that as corollaries of the results above we recover the corresponding results for finite Rees index semigroups, and those for subgroups of groups with finite index. In this section we will mention a number of other applications of the main results.

6.1 Removing an irregular $\mathcal{D}$-class

For definitions and background regarding regular semigroups we refer the reader to [9]. Let S be a semigroup, and let D be a $\mathcal{D}$-class of S such that D is a union of finitely many $\mathcal{R}^S$- and $\mathcal{L}^S$-classes and $T = S \setminus D$ is a subsemigroup of S. In general the removal of D from S to obtain T can have a major effect in terms of properties S and T share. Consider the following example. Let $S \leq T_{\mathbb{Z}}$ the subsemigroup of the full transformation semigroup generated by the set $\{\alpha, \alpha^{-1}, \beta\}$ where

$$\alpha = \begin{pmatrix} \dots & -2 & -1 & 0 & 1 & 2 & \dots \end{pmatrix}, \quad \beta = \begin{pmatrix} \dots & -2 & -1 & 0 & 1 & 2 & \dots \\ \dots & 1 & 1 & 1 & 1 & 1 & \dots \end{pmatrix}.$$

Let $D = \langle \alpha, \alpha^{-1} \rangle$, the group of units of S, and let $T = S \setminus D$. Then D is a $\mathcal{D}$-class of S with a single $\mathcal{R}$- and $\mathcal{L}$-class. However, S and T are quite different in terms of algebraic properties. For instance: S is finitely generated, T is not; T is locally finite, S is not.

On the other hand, if the $\mathcal{D}$-class D happens not to be regular then we do obtain a positive result.

Proposition 35. *Let S be a semigroup, and let D be a $\mathcal{D}$-class of S which is a union of finitely many $\mathcal{R}^S$- and $\mathcal{L}^S$-classes and such that $T = S \setminus D$ is a subsemigroup of S. If D is not regular then T has finite Green index in S.*

Proof. Since D is not regular it follows that the $\mathcal{R}^T$-classes of D coincide with the $\mathcal{R}^S$-classes of D, and similarly for the $\mathcal{L}^T$-classes. Therefore there are only finitely many of each. $\square$

A concrete example where this occurs is the following. Let $S = \mathcal{PF}(\mathbb{Z})$ the finitary power semigroup of the additive group $\mathbb{Z}$. So the elements of S are the finite subsets of $\mathbb{Z}$ and multiplication is given by $AB = A + B$. For $i \in \mathbb{N}$ let $D_i = \{\{a, a+i\} : a \in \mathbb{Z}\}$. These are all of the $\mathcal{D}$-classes of S containing sets with just two elements. Note that since S is commutative it follows that $\mathcal{R} = \mathcal{L} = \mathcal{D}$. The proposition now tells us that we can remove any finite number of these $\mathcal{D}$-classes D_i and it will leave us with a subsemigroup of S that has finite Green index.

6.2 Inverse semigroups

For definitions and background regarding inverse semigroups we refer the reader to [15]. Let S be an inverse semigroup and let ρ be a congruence on S. Let $K = \bigcup_{e \in E} e/\rho$, which is the union of all idempotent ρ-classes of S. Then K is a full inverse subsemigroup of S, in the sense that it contains all of the idempotents. For each ρ-class C of S we let $F(C) = \{c^{-1}c : c \in C\}$ which is a subset of $E(S)$ and therefore of K. We say that K is *finitely covered* if for each class C the set $F(C)$ has finitely many maximal elements (under the natural partial ordering of idempotents in S) and every element of $F(C)$ lies underneath at least one such element. In [2] the above situation is studied in some detail. In particular it is proved that, with the above hypotheses, S has property $\mathcal{P}$ if and only if K has property $\mathcal{P}$ if $\mathcal{P}$ is any of the following: finitely presented, finitely generated, locally finite and residually finite.

We now provide an example to show that with the above hypotheses, K need not have finite Green index in S.

Example 36. Let S be the bi-cyclic monoid, given by the presentation $\langle\, b, c \mid bc = 1 \,\rangle$. Every element can be written uniquely in the form $c^i b^j$ with $i, j \in \mathbb{N}^0$. Let $T = \{c^i b^j : i + j = 2k \text{ for some } k \in \mathbb{N}^0\}$; note that T is the kernel of the congruence arising from the natural epimorphism from B onto the cyclic group of order 2. Since the idempotents in S form the chain $(\mathbb{N}, \leq)$ it follows that each set $F(C)$ is finitely covered. However, the Green index of T in S is infinite, since there are infinitely many $\mathcal{H}^S$-classes not in T.

If we add the hypothesis that S has finitely many $\mathcal{R}^S$- and $\mathcal{L}^S$-classes then we do obtain a positive result.

Proposition 37. *Let S be an inverse semigroup with finitely many left and right ideals. If ρ is a finite index congruence on S, then the kernel $K = \cup_{e \in E} e/\rho$ of ρ has finite Green index in S.*

Proof. Let $x, y \in S \setminus K$. If $x \mathcal{R}^S y$ and $(x, y) \in \rho$ then

$$y = x(x^{-1}y), \quad x = y(y^{-1}x)$$

where $x^{-1}y, y^{-1}x \in K$. It follows that $x \mathcal{R}^K y$. Since ρ has finite index, and S has finitely many $\mathcal{R}^S$-classes, it follows that $S \setminus K$ has finitely many $\mathcal{R}^K$-classes. A dual argument proves that $S \setminus K$ has finitely many $\mathcal{L}^K$-classes. $\square$

For example, if $S = B(n, G)$ is the Brandt semigroup of degree $n \in \mathbb{N}$ over the group G then the relation $\mathcal{H}$ is a congruence on S and $S/\mathcal{H}$ is finite. Therefore $T = H_1 \cup \ldots \cup H_k \cup \{0\}$, the subsemigroup consisting of the union of all group $\mathcal{H}$-classes of S (including 0), has finite Green index in S.

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